Internal energy and gravitation

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$E_i^{(2)}(|\Psi\rangle)$ requires the addition of some further terms (see I). The calculation of the very same extra terms makes it possible to find a better lower bound. It is easy to derive that

$$\begin{aligned}\epsilon_i^{(2)} &\geqslant B_i^{(2)}(|\Psi\rangle)+\sum_{n<i}(\epsilon_i^{(0)}-\epsilon_n^{(0)})\left(\frac{\epsilon_{i+1}^{(0)}-\epsilon_n^{(0)}}{\epsilon_{i+1}^{(0)}-\epsilon_i^{(0)}}\right)\\ &\quad\times\left(\left|\frac{\langle\Phi_n^{(0)}|\hat{H}_1|\Phi_i^{(0)}\rangle}{\epsilon_n^{(0)}-\epsilon_i^{(0)}}+\langle\Phi_n^{(0)}|\Psi\rangle\right|^2\right)\\ &\geqslant B_i^{(2)}(|\Psi\rangle)\end{aligned} \tag{8}$$

which is the desired final result. It is plainly obvious that if $|\Psi\rangle$ has the form prescribed by any of the corollaries of I, $B_i^{(2)}(|\Psi\rangle)$ itself becomes the better lower bound, for, under these prescribed restrictions, the terms under the summation sign in relation (8) all vanish identically.

It is interesting to note that for $i = 0$ we revert to the ground state and in the configuration space the two lower bounds of relation (8) both become identical with the lower bound derived by Robinson (1969) for the ground state and the configuration space using an entirely different approach and formalism. Our result is completely general and is valid for the ground state as well as the excited states and for any complete inner-product space.

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Abstract. A previous theory of the interaction of an ideal fluid with the gravitational field, as given by Rastall in 1968, is modified. It is now assumed that the natural internal energy per unit mass of the fluid is a function of the natural pressure and density.

In a recent paper (Rastall 1968, to be referred to as II) the interaction of an ideal fluid and a gravitational field was discussed. It was assumed in II, § 6, that $U_E(x)$, the proper internal energy per unit proper mass of the fluid at the point with coordinates x, is a function of the proper pressure $p(x)$ and the proper mass per unit proper volume $\rho(x)$, but is independent of the gravitational potential Φ. (The suffix E in U_E indicates that this quantity is measured in natural units, while the absence of a suffix on p and ρ means that they are measured in Φ_0 units, i.e. the units corresponding to one of the preferred charts of the theory.) We now think that this is not the most natural assumption. We still suppose that U_E is independent of Φ. This follows from the fundamental hypothesis that physical quantities measured in natural units are the same in the potential Φ as in the potential $\Phi+k$ for any constant k, and from the more special assumption† that $U_E(x)$ depends on Φ

† We exclude the possibility that U_E depends on the derivatives of Φ, which it may in strong gravitational fields.

only through its value $\Phi(x)$. However, we now assume that $U_E(x)$ depends on the *natural* quantities $p_E(x)$ and $\rho_E(x)$. It follows that equation (6.10) of II must be replaced by $dU_E = T_E\, d\mathscr{S}_E - p_E\, dv_E$, where $v_E = 1/\rho_E$. We assume that the proper entropy per unit proper mass is dimensionless, and it follows that $\mathscr{S} = \mathscr{S}_E$ (see II, equation(2.3)), and that

$$dU = T\, d\mathscr{S} - p\, dv + c_E^{-2}(4U + 6pv)\, d\Phi. \qquad (6.10')$$

(The modified form of equation (n) of II will be referred to as (n').)

The replacement of (6.10) by (6.10′) entails the following changes in the equations of II:

$$J_{,\mu} = T\mathscr{S}_{,\mu} + \rho^{-1}p_{,\mu} + c_E^{-2}(4J + 2p\rho^{-1})\Phi_{,\mu} \qquad (6.13')$$

$$\frac{DJ_E}{Dt} = \rho_E^{-1}\frac{Dp_E}{Dt} \qquad (6.14')$$

$$\frac{D\theta_\mu}{Dt} = c\gamma^{-1}(u_k\theta_{\mu,k} - s^{-2}u_0\theta_{\mu,0})$$
$$= c_E\gamma^{-1}s^{-3}\{\rho^{-1}p_{,\mu} + c_E^{-2}\Phi_{,\mu}(-J + 2Ju_0^2 + 2p\rho^{-1})\} \qquad (6.18')$$

$$\frac{\partial\mathscr{L}_f}{\partial\Phi} = -c_E^{-2}\left\{\rho J\gamma^2\left(1 + \frac{V^2}{c^2}\right) + 2p\right\} \qquad (6.22')$$

$$\Phi_{,mm} - s^{-4}(\Phi_{,00} - \tfrac{1}{2}(4-\alpha)c_E^{-2}\Phi_{,0}^2) + \tfrac{1}{2}\alpha c_E^{-2}\Phi_{,p}\Phi_{,p}$$
$$= -(2Kc_E^2)^{-1}\left\{\rho J\gamma^2\left(1 + \frac{V^2}{c^2}\right) + 2p\right\}\exp\{-\alpha c_E^{-2}(\Phi - \Phi_1)\}. \qquad (6.23')$$

The other numbered equations of II are unchanged. (Note a misprint in (6.18): the last term should be $-4p\rho^{-1}$.)

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